Announcements

1) Stephen DeBacker returns!

Presentation for math majors, Thursday 4-5

CB 2062

2) HW due tomorrow

Example 1: (partial vs. full pivoting)

$$\begin{bmatrix} 5 & -1 & 4 \\ 12 & 3 & 2 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \\ -9 \end{bmatrix}$$

Concentrate on

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 12 & 3 & 2 \\ 0 & -5 & 4 \end{bmatrix}$$

Full Pivoting

(row & column exchanges)

Find the smallest nonzero absolute value in the first row and first column.

$$\begin{bmatrix} 5 & -1 & 4 \\ 12 & 3 & 2 \\ 0 & -5 & 4 \end{bmatrix}$$

Smallest nonzero absolute value is one (1st row, 2nd column)

Swap first two columns

$$\begin{bmatrix} -1 & 5 & 4 \\ 3 & 12 & 2 \\ -5 & 0 & 4 \end{bmatrix}$$

(apply the permutation matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ on the right

of A)

Add multiples of R1 to R2 and R3.

$$\begin{bmatrix} -1 & 5 & 4 \\ 0 & 27 & 14 \\ 0 & -25 & -16 \end{bmatrix}$$

Smallest absolute value of entries in the lower 2x2 block = 14, 2nd row, 3rd column

Swap Columns 2 & 3

$$\begin{bmatrix} -1 & 4 & 5 \\ 0 & 14 & 27 \\ 0 & -16 & -25 \end{bmatrix}$$

(applying the permutation

matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

on the right)

Add a multiple of R2 to R3

$$\begin{bmatrix} -1 & 4 & 5 \\ 0 & 14 & 27 \\ 0 & 0 & 41/7 \end{bmatrix}$$

Done!

Partial Pivoting

$$\begin{bmatrix} 5 & -1 & 4 \\ 12 & 3 & 2 \\ 0 & -5 & 4 \end{bmatrix}$$

1^{st} stage, no swapping of rows since $5 < 12$.

Add a multiple of R1 to R2

$$\begin{bmatrix} 5 & -1 & 4 \\ 0 & 27/5 & -38/5 \\ 0 & -5 & 4 \end{bmatrix}$$

looking at the 2^{nd} entry of the 2^{nd} & 3^{rd} rows,

$|-5| < |27/5|$, so

Swap R2 with R3

$$\begin{bmatrix} 5 & -1 & 4 \\ 0 & -5 & 4 \\ 0 & 27/5 & -38/5 \end{bmatrix}$$

(multiply by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ on the

left)

Add the correct multiple of R2 to R3

Algorithm for Partial Pivoting

Want $PA = LU$

where A is given,

P is a permutation matrix,

L is lower-triangular,

U is upper-triangular

Algorithm

Initialize

$$A = U, \quad L = P = I_m$$

for $k = 1$ to $m-1$

?

Select $i \geq k$ to maximize $|u_{ik}|$

Switch $U(k, k:m) \rightarrow u(i, k:m)$

Switch $L(k, 1:k-1) \rightarrow L(i, 1:k-1)$

switch $P(k, :) \rightarrow P(i, :)$

(zero flops)

for $j = k+1$ to m

$$L(j,k) = \frac{U(j,k)}{U(k,k)}$$

$$U(j,k:m) = U(j,k:m) - L(j,k)\,U(k,k:m)$$

end

end

Flop Count

for $k=1$ to $m-1$

for $j=k+1$ to m

$$L(j,k)=\frac{U(j,k)}{U(k,k)}$$

$$U(j,k:m)=U(j,k:m)-L(j,k)U(k,k:m)$$

$$\sum_{k=1}^{m-1}\sum_{j=k+1}^{m}\left(1+2(m-k+1)\right)$$

$$\sum_{k=1}^{m-1} \sum_{j=k+1}^{m} \left(1 + 2(m-k+1)\right)$$

$$= \sum_{k=1}^{m-1} (m-k-1)\left(1+2(m-k+1)\right)$$

$$= \sum_{k=1}^{m-1} \left(\begin{array}{l} (m-k) + 2(m-k)(m-k+1) \\ -1 - 2(m-k+1) \end{array} \right)$$

$$= \sum_{k=1}^{m-1} \left(-3 + (m-k) + 2(m-k)^2\right)$$

$$\sum_{k=1}^{m-1} \left(-3 + (m-k) + 2(m-k)^2 \right)$$

$$= \sum_{k=1}^{m-1} \left(-3 + (m-k) + 2m^2 - 4mk + 2k^2 \right)$$

$$= -3(m-1) + m(m-1) - \frac{(m-1)m}{2}$$

$$+ 2m^2(m-1) - \frac{4m(m-1)(m)}{2}$$

$$+ \frac{2m(m-1)(2m-1)}{6} \sim \boxed{\frac{2}{3} m^3 \text{ flops}}$$

Stability

Define

$$\rho = \frac{\displaystyle\max_{\substack{1 \le i \le m \\ 1 \le j \le m}} |A(i,j)|}{\displaystyle\max_{\substack{1 \le i \le m \\ 1 \le j \le m}} |U(i,j)|}$$

Theorem: Gaussian elimination with partial pivoting is backwards stable in the sense that $\exists\ \delta A \in \mathbb{C}^{m \times m}$,

$$\tilde{L}\tilde{U} = \tilde{P}A + \delta A$$

and

$$\frac{\|\delta A\|}{\|A\|} = O(\rho\, \varepsilon_{machine})$$

5x5 example in book:

$$p = 16 = 2^{5-1}.$$

In general, you can find

$$A \in \mathbb{C}^{m \times m}, \quad p(A) = 2^{m-1}.$$